The a_5 heat kernel coefficient on a manifold with boundary

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In this letter we present the calculation of the a_5 heat kernel coefficient of the heat operator trace for a partial differential operator of Laplace type on a compact Riemannian manifold with Dirichlet and Robin boundary conditions.

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Motivated by the need to give answers to some fundamental questions in quantum field theory, during the last years there has been and continues to be a lot of interest in the problem of calculating the heat kernel coefficients of partial differential operators (mostly of Laplace type) (see, for example [1–5]). The coefficients contain information about the scaling and divergence behaviour of the quantum field theory. Furthermore, they determine effective actions at high temperature [6–8] or for large masses of the fields involved [9] and they provide expansions in different background fields [10]. In mathematics the interest in the heat kernel coefficients stems, in particular, from the well known connection that exists between the heat equation and the Atiyah-Singer index theorem [11].

In general, if the manifold $\mathcal{M}$ (which we assume to be a compact Riemannian one) has a boundary $\partial\mathcal{M}$, the coefficients a_n in the short-time expansion have a volume and a boundary part. For the volume part very effective systematic schemes have been developed (see for example [10,12,13]). The calculation of the boundary part is in general more difficult. Only quite recently has the coefficient a_4 for Dirichlet and Robin boundary conditions been found [14,15] (see also [16–22]).

For the a_5 coefficient only partial results are available [23]. To state them in detail let us introduce some notation (see also [23], but with characteristic differences). Let V be a smooth vector bundle over $\mathcal{M}$ equipped with a connection ∇^V and E an endomorphism of V. Define the partial differential operator

$$D = -g^{ij}\nabla_i^V\nabla_j^V - E. \tag{1}$$

Furthermore we must impose suitable boundary conditions. If $\phi \in C^\infty(V)$, let $\phi_{;m}$ be the covariant derivative with respect to the exterior unit normal, let S be an endomorphism of $V|_{\partial\mathcal{M}}$. We define the Dirichlet boundary operator $\mathcal{B}^-$ and the Robin boundary operator B_S^+ by

$$\mathcal{B}^-\phi \equiv \phi|_{\partial\mathcal{M}} \quad \text{and} \quad \mathcal{B}_S^+\phi \equiv (\phi_{;m} - S\phi)\,|_{\partial\mathcal{M}}. \tag{2}$$

To have a uniform notation we set $S=0$ for Dirichlet boundary conditions and write $B_S^\mp$. Let $D_\mathcal{B}$ be the operator defined by the appropriate boundary conditions. If F is a smooth function on $\mathcal{M}$, there is an asymptotic series as $t \to 0$ of the form

$$\mathrm{Tr}_{L^2}\left(Fe^{-tD_\mathcal{B}}\right) \approx \sum_{n\geq 0} t^{\frac{n-m}{2}} a_n(F,D,\mathcal{B}), \tag{3}$$

where the $a_n(F,D,\mathcal{B})$ are locally computable [24] .

We are now ready to state the results of [23]. The coefficient a_5 for an arbitrary smearing function F has been calculated for totally geodesic boundary $\partial\mathcal{M}$. Furthermore, putting $F=1$ it has been found for $\mathcal{M}$ being a domain in $\mathbb{R}^m$. Finally it has been shown that for a smooth but not necessarily totally geodesic boundary, there exist universal constants so that

$$\begin{aligned}
&a_5(F,D,B_S^\mp) = \mp 5760^{-1}(4\pi)^{-(m-1)/2}\mathrm{Tr}\{F\left\{g_1E_{;mm} + g_2E_{;m}S + g_3E^2\right.\\
&+g_4E_{:a}{}^a + g_5RE + 120\Omega_{ab}\Omega^{ab} + g_6\Delta R + g_7R^2 + g_8R_{ij}R^{ij} + g_9R_{ijkl}R^{ijkl}\\
&+g_{10}R_{mm}E + g_{11}R_{mm}R + g_{12}RS^2 + (-360^-, 90^+)\Omega_{am}\Omega^a{}_m + g_{13}R_{;mm} + g_{14}R_{mm:a}{}^a\\
&+g_{15}R_{mm;mm} + g_{16}R_{;m}S + g_{17}R_{mm}S^2 + g_{18}SS_{:a}{}^a + g_{19}S_{:a}S_:^a + g_{20}R_{ammb}R^{ab}\\
&\left.+g_{21}R_{mm}R_{mm} + g_{22}R_{ammb}R^a{}_{mm}{}^b + g_{23}ES^2 + g_{24}S^4\right\}
\end{aligned}$$

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$$
\begin{aligned}
&+F_{;m}\left\{g_{25}R_{;m}+g_{26}RS+g_{27}R_{mm}S+g_{28}S_{:a}{}^{a}+g_{29}E_{;m}+g_{30}ES+g_{31}S^3\right\}\\
&+F_{;mm}\left\{g_{32}R+g_{33}R_{mm}+g_{34}E+g_{35}S^2\right\}+g_{36}SF_{;mmm}+g_{37}F_{;mmmm}\\
&+F\left\{d_1KE_{;m}+d_2KR_{;m}+d_3K^{ab}R_{ammb;m}+d_4KS_{:b}{}^{b}+d_5K_{ab}S^{ab}_{:}\right.\\
&+d_6K_{:b}S^b_{:}+d_7K_{ab:}{}^{a}S^b_{:}+d_8K_{:b}{}^{b}S+d_9K_{ab:}{}^{ab}S+d_{10}K_{:b}K^b_{:}+d_{11}K_{ab:}{}^{a}K^b_{:}\\
&+d_{12}K_{ab:}{}^{a}K^{bc}{}_{:c}+d_{13}K_{ab:c}K^{ab}{}_{:}{}^{c}+d_{14}K_{ab:c}K^{ac}{}_{:}{}^{b}+d_{15}K_{:b}{}^{b}K\\
&+d_{16}K_{ab:}{}^{ab}K+d_{17}K_{ab:}{}^{a}{}_{c}K^{bc}+d_{18}K_{:bc}K^{bc}+d_{19}K_{bc:a}{}^{a}K^{bc}\\
&+g_{38}KSE+d_{20}KSR_{mm}+g_{39}KSR+d_{21}K_{ab}R^{ab}S+d_{22}K^{ab}SR_{ammb}+g_{40}K^2E\\
&+g_{41}K_{ab}K^{ab}E+g_{42}K^2R+g_{43}K_{ab}K^{ab}R+d_{23}K^2R_{mm}\\
&+d_{24}K_{ab}K^{ab}R_{mm}+d_{25}KK_{ab}R^{ab}+d_{26}KK^{ab}R_{ammb}+d_{27}K_{ab}K^{ac}R^b_c\\
&+d_{28}K^b_aK^{ac}R_{bmmc}+d_{29}K_{ab}K_{cd}R^{acbd}+d_{30}KS^3+d_{31}K^2S^2+d_{32}K_{ab}K^{ab}S^2\\
&+d_{33}K^3S+d_{34}KK_{ab}K^{ab}S+d_{35}K_{ab}K^{bc}K^a_cS+d_{36}K^4+d_{37}K^2K_{ab}K^{ab}\\
&\left.+d_{38}K_{ab}K^{ab}K_{cd}K^{cd}+d_{39}KK_{ab}K^{bc}K^a_c+d_{40}K_{ab}K^{bc}K_{cd}K^{da}\right\}\\
&+F_{;m}\left\{g_{44}KE+d_{41}KR_{mm}+g_{45}KR+d_{42}KS^2\right.\\
&+d_{43}K_{:b}{}^{b}+d_{44}K_{ab:}{}^{ab}+d_{45}K_{ab}R^{ab}+d_{46}K^{ab}R_{ammb}+d_{47}K^2S\\
&\left.+d_{48}K_{ab}K^{ab}S+d_{49}K^3+d_{50}KK_{ab}K^{ab}+d_{51}K_{ab}K^{bc}K^a_c\right\}\\
&+F_{;mm}\left\{d_{52}KS+d_{53}K^2+d_{54}K_{ab}K^{ab}\right\}+d_{55}KF_{;mmm}\}[\partial\mathcal{M}]
\end{aligned}
\tag{4}
$$

Here and in the following $f[\mathcal{M}]=\int_{\mathcal{M}}dx\,f(x)$ and $f[\partial\mathcal{M}]=\int_{\partial\mathcal{M}}dyf(y)$, with dx and dy being the Riemannian volume elements of $\mathcal{M}$ and $\partial\mathcal{M}$. In addition, ";" denotes differentiation with respect to the Levi-Civita connection of $\mathcal{M}$ and ":" covariant differentiation tangentially with respect to the Levi-Civita connection of the boundary. Finally, our sign convention is $R^i{}_{jkl}=-\Gamma^i_{jk,l}+\Gamma^i_{jl,k}+\Gamma^i_{nk}\Gamma^n_{jl}-\Gamma^i_{nl}\Gamma^n_{jk}$ (see for example [25]).

In addition to the terms containing the curvature Ω of the connection ∇^V, in [23] the values of $g_1,...,g_{45}$ and d_{43},d_{44} and d_{55} were calculated. However, the main group of terms containing the extrinsic curvature K_{ab} and its derivatives remained undetermined and will be found here for the first time. In order to explain the way we proceed we will take $g_1,...,g_{45}$ and $d_1,...,d_{55}$ as unknown. The small group containing the curvature Ω is completely known and will have no influence on the calculation presented here.

We are going to use essentially three different ingredients. The first one is a Lemma on product manifolds [14]. Let $N^\nu(F)=F_{;m...}$ be the ν^{th} normal covariant derivative. There exist local formulae $a_n(x,D)$ and $a_{n,\nu}(y,D)$ so that

$$a_n(F,D,\mathcal{B}_S^{\mp})=\{Fa_n(x,D)\}[\mathcal{M}]+\{\sum_{\nu=0}^{2n-1}N^\nu(F)a_{n,\nu}(y,D,\mathcal{B}_S^{\mp})\}[\partial\mathcal{M}].$$

Let $\mathcal{M}=\mathcal{M}_1\times\mathcal{M}_2$ and $D=D_1\otimes 1+1\otimes D_2$ and $\partial\mathcal{M}_2=\emptyset$. Then

$$a_{n,\nu}(y,D,\mathcal{B}_S^{\mp})=\sum_{p+q=n}a_{p,\nu}(y_1,D_1,\mathcal{B}_S^{\mp})a_q(x_2,D_2).$$

For a_5 this means

$$a_5(y,D,\mathcal{B}_S^{\mp})-a_5(y_1,D_1,\mathcal{B}_S^{\mp})a_0(x_2,D_2)=a_3(y_1,D_1,\mathcal{B}_S^{\mp})a_2(x_2,D_2)+a_1(y_1,D_1,\mathcal{B}_S^{\mp})a_4(x_2,D_2). \tag{5}$$

This gives the following 22 universal constants:

$g_3=720$	$g_5=240$	$g_6=48$	$g_7=20$
$g_8=-8$	$g_9=8$	$g_{10}=-120$	$g_{11}=-20$
$g_{12}=480$	$g_{23}=2880$	$g_{26}=-240$	$g_{30}-1440$
$g_{32}=60$	$g_{34}=360$	$g_{38}=1440$	$g_{39}=240$
$g_{40}=(105^-,195^+)$	$g_{41}=(-150^-,30^+)$	$g_{42}=(105/6^-,195/6^+)$	$g_{43}=(-25^-,5^+)$
$g_{44}=(450^-,-90^+)$	$g_{45}=(75^-,-15^+)$		

(6)

Next we use the calculations on the m-dimensional ball. For $F=1$ the results are given in [26,27], the generalization to arbitrary F has been achieved recently and will be presented in [28]. This leads to the following additional 25 constants or relations among them:

$$
\begin{array}{lll}
g_{24}=1440 & g_{31}=-720 & g_{35}=360 \\
g_{36}=-180 & g_{37}=45 & d_{30}=2160 \\
d_{31}=1080 & d_{32}=360 & d_{33}=885/4 \\
d_{34}=315/2 & d_{35}=150 & d_{36}=(-65/128^-,2041/128^+) \\
d_{37}=(-141/32^-,417/32^+) & d_{40}=(-327/8^-,231/8^+) & d_{42}=-600 \\
d_{47}=-705/4 & d_{48}=75/2 & d_{49}=(495/32^-,-459/32^+) \\
d_{50}=(-1485/16^-,-267/16^+) & d_{51}=(225/2^-,54^+) & d_{52}=30 \\
d_{53}=(1215/16^-,315/16^+) & d_{54}=(-945/8^-,-645/8^+) & d_{55}=(105^-,30^+)
\end{array} \tag{7}
$$

and $d_{38}+d_{39}=(1049/32^-,1175/32^+)$.

All this information is now a very good starting point to use relations of the heat kernel coefficients under conformal rescalings [14]. The relevant ones for our case read

$$
\frac{d}{d\epsilon}|_{\epsilon=0}a_5\left(1,e^{-2\epsilon F}D)\right)-(m-5)a_5(F,D)=0 \tag{8}
$$

$$
\frac{d}{d\epsilon}|_{\epsilon=0}a_5\left(e^{-2\epsilon f}F,e^{-2\epsilon f}D\right)=0 \quad \text{for } m=7. \tag{9}
$$

Setting to zero the coefficients of all terms in (8) we obtain the following equations. (They are ordered in a way such that nearly every equation immediately yields a universal constant. This was the main motivation for the given ordering.)

$$
\begin{array}{ll}
\underline{\text{Term}} & \underline{\text{Coefficient}} \\
EF_{;mm} & 0=-2g_1+(m-2)g_3-2(m-1)g_5-(m-1)g_{10}-(m-5)g_{34} \\
ESF_{;m} & 0=-2g_2-(m-2)g_{23}+(m-1)g_{38}-(m-5)g_{30} \\
SF_{;mmm} & 0=\frac12(m-2)g_2-2(m-1)g_{16}-(m-5)g_{36} \\
KSF_{;mm} & 0=\frac12(m-2)g_2-2(m-1)g_{16}+\frac12(m-2)g_{38}-(m-1)d_{20}-2(m-1)g_{39}-d_{21}+d_{22}-(m-5)d_{52} \\
FE_{:a}{}^{a} & 0=-g_1+(m-2)g_3-(m-5)g_4-2(m-1)g_5-g_{10} \\
F_{;mmmm} & 0=\frac12(m-2)g_1-2(m-1)g_6-2(m-1)g_{13}-(m-1)g_{15}-(m-5)g_{37} \\
F\Delta R & 0=\frac12(m-2)g_5-(m-4)g_6-4(m-1)g_7-mg_8-4g_9-g_{11}-g_{13}+\frac12 g_{20} \\
FR_{;mm} & 0=-\frac12(m-2)g_5+(m-4)g_6+4(m-1)g_7+2(m-1)g_8+8g_9+g_{11}+g_{13}-2g_{15}-\frac{m}{2}g_{20}+g_{22} \\
FR_{mm:a}{}^{a} & 0=\frac12(m-2)g_1-2(m-1)g_6+\frac12(m-2)g_{10}-2(m-1)g_{11}-2(m-1)g_{13}-(m-5)g_{14}-2g_{15} \\
 & \quad +(m-1)g_{20}-2g_{21}-2g_{22} \\
F_{;mm}S^2 & 0=-2(m-1)g_{12}-(m-1)g_{17}+\frac12(m-2)g_{23}-(m-5)g_{35} \\
FS_{:a}S^{a}_{:} & 0=-4(m-1)g_{12}-2g_{17}-(m-3)g_{18}+2g_{19}+(m-2)g_{23} \\
F_{;m}E_{;m} & 0=-5g_1-\frac12(m-2)g_2+(m-1)d_1-(m-5)g_{29} \\
F_{;mmm}K & 0=\frac12(m-2)g_1-4(m-1)g_6-2(m-1)g_{13}-g_{15}+\frac12(m-2)d_1-2(m-1)d_2+d_3-(m-5)d_{55} \\
F_{;m}R_{;m} & 0=-\frac14(m-2)g_1+(2m-7)g_6+(m-6)g_{13}-2g_{15}-\frac12(m-2)g_{16}+(m-1)d_2-\frac12 d_3-(m-5)g_{25} \\
F_{;mm}R_{mm} & 0=-(m-2)g_1+4(m-1)g_6-2(m-2)g_8-8g_9+\frac12(m-2)g_{10}-2(m-1)g_{11} \\
 & \quad +4(m-1)g_{13}-2(m-1)g_{21}-2g_{22}-(m-5)g_{33} \\
F_{;m}R_{mm}S & 0=-\frac12(m-2)g_2+2(m-1)g_{16}-(m-2)g_{17}+(m-1)d_{20}-d_{21}-d_{22}-(m-5)g_{27} \\
FKS_{:a}{}^{a} & 0=-(m-4)d_4-d_5+d_6+d_7-d_8-d_9+\frac12(m-2)g_{38}-d_{20}-2(m-1)g_{39}-d_{21} \\
FK_{:a}{}^{a}S & 0=-\frac12(m-2)g_2+2(m-1)g_{16}-d_4+d_6-(m-4)d_8+\frac12(m-2)g_{38}-d_{20}-2(m-1)g_{39}-d_{21} \\
FK_{ab}S^{ab}_{:} & 0=-(m-2)g_2+4(m-1)g_{16}+3d_5-(m-2)d_7+(m-2)d_9-(m-2)d_{21}+d_{22} \\
FK_{ab:}{}^{ab}S & 0=-d_5+d_7-(m-4)d_9-(m-2)d_{21}+d_{22} \\
F_{;m}S_{:a}{}^{a} & 0=\frac12(m-2)g_2-2(m-1)g_{16}-(m-2)g_{18}+(m-2)g_{19}+(m-1)d_4+d_5 \\
 & \quad -(m-1)d_6-d_7+(m-1)d_8+d_9-(m-5)g_{28}
\end{array} \tag{10}
$$

The equations given up to this point allow for the determination of the universal constants apart from two groups. The first group is $d_{23},...,d_{29},d_{38},d_{39},d_{41},d_{45},d_{46}$. The second one $d_{10},...,d_{19},d_{43},d_{44}$. Explicitly we obtained

$$
\begin{array}{lll}
g_1=360 & g_2=-1440 & g_4=240 \\
g_{13}=12 & g_{14}=24 & g_{15}=15 \\
g_{16}=-270 & g_{17}=120 & g_{18}=960 \\
g_{19}=600 & g_{20}=-16 & g_{21}-17 \\
g_{22}=-10 & g_{25}=(60^-,195/2^+) & g_{27}=90 \\
g_{28}=-270 & g_{29}=(450^-,630^+) & g_{33}=-90 \\
d_1=(450^-,-90^+) & d_2=(42^-,-111/2^+) & d_3=(0^-,30^+) \\
d_4=240 & d_5=420 & d_6=390 \\
d_7=480 & d_8=420 & d_9=60 \\
d_{20}=30 & d_{21}=-60 & d_{22}=-180
\end{array} \tag{11}
$$

The first group is completely determined using the relations

$$
\begin{array}{ll}
\underline{\text{Term}} & \underline{\text{Coefficient}} \\
F_{;mm}K_{ab}K^{ab} & 0 = -(m-2)g_1 + 4(m-1)g_6 + 4(m-1)g_{13} + 2g_{15} + d_3 + \frac{1}{2}(m-2)g_{41} \\
 & \quad - 2(m-1)g_{43} - (m-1)d_{24} - d_{27} + d_{28} - (m-5)d_{54} \\
F_{;mm}K^2 & 0 = -2(m-1)g_6 + \frac{1}{2}(m-2)d_1 - 2(m-1)d_2 + \frac{1}{2}(m-2)g_{40} - 2(m-1)g_{42} \\
 & \quad - (m-1)d_{23} - d_{25} + d_{26} - (m-5)d_{53} \\
F_{;m}KR & 0 = \frac{1}{2}(m-2)g_5 - 2g_6 - 4(m-1)g_7 - 2g_8 - g_{11} - 2d_2 - \frac{1}{2}(m-2)g_{39} \\
 & \quad + 2(m-1)g_{42} + 2g_{43} + d_{25} - (m-5)g_{45} \\
F_{;m}KR_{mm} & 0 = \frac{1}{2}(m-2)g_1 + \frac{1}{2}(m-2)g_{10} - 2(m-1)g_{11} - 2(m-1)g_{13} + 4g_{15} + g_{20} \\
 & \quad - 2g_{21} - \frac{1}{2}(m-2)d_1 + 2(m-1)d_2 + d_3 - \frac{1}{2}(m-2)d_{20} + 2(m-1)d_{23} \\
 & \quad + 2d_{24} - d_{25} - d_{26} - (m-5)d_{41} \\
F_{;m}K_{ab}R^{ab} & 0 = -\frac{1}{2}(m-2)g_1 + 2(m-1)g_6 - 2(m-2)g_8 - 8g_9 + 2(m-1)g_{13} - 4g_{15} + g_{20} \\
 & \quad - d_3 - \frac{1}{2}(m-2)d_{21} + (m-1)d_{25} + 2d_{27} + 2d_{29} - (m-5)d_{45} \\
F_{;m}K^{ab}R_{ammb} & 0 = -(m-2)g_1 + 4(m-1)g_6 + 4(m-1)g_{13} + 2g_{15} - (m-2)g_{20} + 2g_{22} - d_3 \\
 & \quad - \frac{1}{2}(m-2)d_{22} + (m-1)d_{26} + 2d_{28} + 2d_{29} - (m-5)d_{46} \\
F_{;m}K_{ab}K^{bc}K_c^a & 0 = (m-2)g_1 - 4(m-1)g_6 - 4(m-1)g_{13} - 2g_{15} - d_3 - (m-2)d_{27} + d_{28} \\
 & \quad + 2d_{29} - \frac{1}{2}(m-2)d_{35} + (m-1)d_{39} + 4d_{40} - (m-5)d_{51} \\
FR_{ac}K_b^cK^{ab} & 0 = -2(m-2)g_8 - 8g_9 + 4g_{15} + g_{20} + 2d_3 + 4d_{13} + 4d_{14} - 4d_{19} - (m-2)d_{27} + d_{28} + 2d_{29}
\end{array}
\tag{12}
$$

One finds

$$
\begin{array}{lll}
d_{23} = (-215/16^-, -275/16^+) & d_{24} = (-215/8^-, -275/8^+) & d_{25} = (14^-, -1^+) \\
d_{26} = (-49/4^-, -109/4^+) & d_{27} = 16 & d_{28} = (47/2^-, -133/2^+) \\
d_{29} = 32 & d_{38} = (777/32^-, 375/32^+) & d_{39} = (17/2^-, 25^+) \\
d_{41} = (-255/8^-, 165/8^+) & d_{45} = (-30^-, -15^+) & d_{46} = (-465/4^-, -165/4^+)
\end{array}
\tag{13}
$$

Finally let us consider the second group mentioned above. As we will see, one needs just one more relation in addition to those obtained from equation (8), which are, in detail,

$$
\begin{array}{ll}
\underline{\text{Term}} & \underline{\text{Coefficient}} \\
FK_{:b}K^{:b} & 0 = 2(m-1)g_6 - 4g_{15} - (m-2)g_{20} + 2g_{22} - \frac{1}{2}(m-2)d_1 + 2(m-1)d_2 + 2d_{10} \\
 & \quad + d_{11} - (m-3)d_{15} - d_{16} - d_{18} + (m-2)g_{40} - 4(m-1)g_{42} - 2d_{23} - 2d_{25} \\
FK_{ab:}{}^{a}K^{b}_{:} & 0 = 2(m-2)g_1 - 4(m-1)g_6 - 8(m-1)g_{13} + (4m-6)g_{20} - 8g_{22} - (m-2)d_1 \\
 & \quad + 4(m-1)d_2 - (m-3)d_{11} + 2d_{12} - 2d_{14} + 2d_{16} - 2d_{17} + 2d_{18} - 2(m-2)d_{25} + 2d_{26} - 4d_{29} \\
FK_{ab:c}K^{ab}{}_{:}{}^{c} & 0 = (m-2)g_1 - 4(m-1)g_6 - 4(m-1)g_{13} - 2g_{15} + (m-2)g_{20} - 2g_{22} - 3d_3 \\
 & \quad + 2d_{13} + 2d_{14} - (m-3)d_{19} + (m-2)g_{41} - 4(m-1)g_{43} - 2d_{24} - 2d_{27} \\
FKK_{ab:}{}^{ab} & 0 = 4(m-2)g_8 + 16g_9 - 4g_{15} - mg_{20} + 2g_{22} + d_{11} + 2d_{12} - (m-4)d_{16} \\
 & \quad - 2d_{17} - d_{18} - (m-2)d_{25} + d_{26} - 2d_{29} \\
F_{;m}K_{:a}{}^{a} & 0 = -\frac{3}{2}(m-2)g_1 + 4(m-1)g_6 - 4(m-2)g_8 - 16g_9 + 6(m-1)g_{13} + \frac{1}{2}(m-2)d_1 \\
 & \quad - 2(m-1)d_2 - d_3 - \frac{1}{2}(m-2)d_4 + \frac{1}{2}(m-2)d_6 - \frac{1}{2}(m-2)d_8 - 2(m-1)d_{10} \\
 & \quad - d_{11} - 2d_{13} + 2(m-1)d_{15} + d_{16} + d_{18} + 2d_{19} - (m-5)d_{43} \\
F_{;m}K_{ab:}{}^{ab} & 0 = \frac{1}{2}(m-2)g_1 - 2(m-1)g_6 + 4(m-2)g_8 + 16g_9 - 2(m-1)g_{13} + 2g_{15} + 2d_3 \\
 & \quad - \frac{1}{2}(m-2)d_5 + \frac{1}{2}(m-2)d_7 - \frac{1}{2}(m-2)d_9 - (m-1)d_{11} - 2d_{12} - 2d_{14} \\
 & \quad + (m-1)d_{16} + 2d_{17} + (m-1)d_{18} - (m-5)d_{44} \\
FK_{ab:}{}^{a}K^{bc}{}_{:c} & 0 = (4-3m)g_{20} + 6g_{22} - 2d_3 - 2(m-2)d_{12} - 4d_{13} - 2d_{14} \\
 & \quad + (m+1)d_{17} + 4d_{19} - (m-2)d_{27} + d_{28} + 2d_{29} \\
FK_{:ab}K^{ab} & 0 = 2(m-2)g_1 - 4(m-1)g_6 - 2(m-2)g_8 - 8g_9 - 8(m-1)g_{13} + (4m-5)g_{20} \\
 & \quad - 8g_{22} - (m-2)d_1 + 4(m-1)d_2 - (m-2)d_{11} - 2d_{14} + (m-2)d_{16} + 3d_{18} \\
 & \quad - (m-2)d_{25} + d_{26} - 2d_{29}
\end{array}
\tag{14}
$$

This yields

$$
\begin{array}{ll}
d_{11} = (58^-, 238^+) & d_{15} = (6^-, 111^+) \\
d_{16} = (-30^-, -15^+) & d_{19} = (54^-, 114^+)
\end{array}
\tag{15}
$$

together with the relations

$$
\begin{array}{ll}
2d_{10} + d_{43} = -91 & 2d_{10} - d_{18} = (-983/8^-, -1403/8^+) \\
2d_{14} - 3d_{18} = (-913/4^-, -2533/4^+) & d_{13} + d_{14} = (297/8^-, 837/8^+) \\
d_{18} - d_{44} = (60^-, 225^+) & 2d_{12} - 2d_{17} - d_{18} = (-7/4^-, -787/4^+) \\
2d_{12} - d_{17} = 32 &
\end{array}
\tag{16}
$$

This is all we can get with the relation (8). It is seen, that for example given d_{43} or d_{44} the remaining constants may be determined. This is achieved with the equation (9) [23]. Thus at the end one gets

$$\begin{array}{lll} d_{10} = (-413/16^-, 487/16^+) & d_{12} = (-11/4^-, 49/4^+) & d_{13} = (355/8^-, 535/8^+) \\ d_{14} = (-29/4^-, 151/4^+) & d_{17} = (-75/2^-, -15/2^+) & d_{18} = (285/4^-, 945/4^+) \\ d_{43} = (-315/8^-, -1215/8^+) & d_{44} = 45/4 & \end{array} \tag{17}$$

In summary, we have determined the full a_5 heat-kernel coefficient for Dirichlet and Robin boundary conditions. All terms not displayed in the above lists have been used as a check for the universal constants.

We have shown that special case evaluation of heat kernel coefficients supplemented by the conformal techniques developed in [14] provide a very powerful tool for the calculation of the coefficients on general manifolds. The techniques displayed might prove very useful in finding universal constants for recently discussed generalized boundary conditions [29–34] having relevance for one-loop quantum gravity and gauge theory. However, in these cases the task will be even more difficult due to the many additional possibilities of building geometrical invariants [31,34].

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[1] Ramond P. *Field Theory. A Modern Primer.* Massachusetts, (1981). The Benjamin/Cummings Publishing Company, INC.
[2] Birrell N and Davies P C W. *Quantum Fields in Curved Spaces.* Cambridge, (1982). Cambridge University Press.
[3] Buchbinder I L and Odintsov S D and Shapiro I L. *Effective Action in Quantum Gravity.* Bristol and Philadelphia, (1992). IOP Publishing.
[4] Fulling S A., *Aspects of Quantum Field Theory in Curved Space-Time*, Cambridge (1989) Cambridge University Press.
[5] Esposito G., *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries.* Lecture Notes in Physics, Monographs, Vol. m12, Berlin: Springer-Verlag, 1994.
[6] Dowker J S and Schofield J P. Nucl. Phys. B, **327**, 267 (1989).
[7] Toms D J. Phys. Rev. Lett., **8**, 1152 (1992).
[8] Bytsenko A A, Vanzo L and Zerbini S. Phys. Lett. B, **291**, 26 (1992).
[9] Ball R and Osborn H. Nucl. Phys. B, **263**, 245 (1986).
[10] Avramidi I G. Nucl. Phys. B, **355**, 712 (1991).
[11] Gilkey P B. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* 2nd. Edn., CRC Press, Boca Raton 1995.
[12] Fulling S A and Kennedy G. Transac. Amer. Math. Soc., **310**, 583 (1988).
[13] Amsterdamski P, Berkin A L, and O'Connor D J. Class. Quantum Grav., **6**, 1981 (1989).
[14] Branson T P and Gilkey P B. Commun. PDE, **15**, 245 (1990).
[15] Vassilevich D V. J. Math. Phys. **36** 3174 (1995).
[16] Moss I G and Dowker J S. Phys. Lett. B, **229**, 261 (1989).
[17] Mc Avity D M and Osborn H. Class. Quantum Grav., **8**, 603 (1991).
[18] Dettki A and Wipf A. Nucl. Phys. B, **377**, 252 (1992).
[19] Dowker J S and Schofield J P. J. Math. Phys., **31**, 808 (1990).
[20] Cognola G, Vanzo L, and Zerbini S. Phys. Lett. B, **241**, 381 (1990).
[21] D'Eath P D and Esposito G. Phys. Rev. D, **43**, 3234 (1991).
[22] Moss I G and Poletti S. Phys. Lett. B, **333**, 326 (1994).
[23] Branson T P, Gilkey P B and Vassilevich D V., The Asymptotics of the Laplacian on a Manifold with Boundary II, hep-th/9504029.
[24] Grubb G., Functional calculus of pseudo differential boundary problems, Progress in Math. 65, Birkhäuser, Boston, 1986.
[25] Hawking S W and Ellis G F R., The large scale structure of space-time, Cambridge University Press, Cambridge, 1973.
[26] Bordag M, Elizalde E and Kirsten K., J. Math. Phys. **37**, 895 (1996).
[27] Bordag M, Kirsten K and Dowker S., Comm. Math. Phys. **182**, 371 (1996).

[28] Dowker S and Kirsten K., in preparation.
[29] Barvinsky A O., Phys. Lett. B, **195**, 344 (1987).
[30] McAvity D M and Osborn H., Class. Quantum Grav. **8**, 1445 (1991).
[31] Avramidi I G and Esposito G., Heat kernel asymptotics with generalised boundary conditions, hep-th/9701018.
[32] Moss I G and Silva P J., Phys. Rev. D **55**, 1072 (1997).
[33] Dowker J S and Kirsten K., Heat-kernel coefficients for oblique boundary conditions, hep-th/9706129, to appear in Class. Quantum Grav.
[34] Avramidi I G and Esposito G., The $A_{5/2}$ coefficient in heat-kernel asymptotics, in preparation.